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The Formulation of Invariant Imbedding Method to Solve Multipoint Discrete Boundary Value Problems

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Abstract. In this paper we shall formulate the invariant imbedding technique to solve linear discrete systems satisfying multipoint boundary conditions. An application to potential equation is also illustrated.

1. INTRODUCTION

The purpose of this paper is to formulate invariant imbedding method to compute the solution of the linear discrete system

$$\mathcal{U}(k+1) = \mathcal{A}(k)\mathcal{U}(k) + \mathcal{B}(k), \quad k \in N(k_1, k_r) \tag{1.1}$$

satisfying the multipoint boundary conditions

$$\sum_{i=1}^{r} L^i \mathcal{U}(k_i) = \mathcal{L}, \tag{1.2}$$

where $\mathcal{A}(k)$ is a given nonsingular $n \times n$ matrix with elements $a_{ij}(k)$, $1 \le i, j \le n$; $\mathcal{B}(k)$ is a given $n \times 1$ vector with components $b_i(k), 1 \le i \le n$; $\mathcal{U}(k)$ is an unknown $n \times 1$ vector with components $u_i(k), 1 \le i \le n$; $0 \le k_1 < k_2 < \cdots < k_r (r \ge 2)$ are positive integers; $N(k_1, k_r)$ is the discrete interval $[k_1, k_1 + 1, \ldots, k_r]$; L^s, $1 \le s \le r$ are given $n \times n$ matrices with elements α^s_{ij}, $1 \le i, j \le n$ and $\mathcal{L}$ is the known $n \times 1$ vector with components ℓ_i, $1 \le i \le n$.

In recent years the above problem has been a subject of considerable study, especially because various discretizations of continuous boundary value problems lead to particular cases of (1.1), (1.2), e.g., see [1]. Necessary and sufficient conditions ensuring the existence and uniqueness of solutions of (1.1), (1.2) are available in [2, also see references therein]; whereas the method of complementary functions and the method of particular solutions, the method of adjoints, and the method of chasing have been proposed and illustrated by solving several numerically unstable problems in [3–6], respectively. The well known [7,8] invariant imbedding technique for continuous problems has also been formulated for solving two point discrete boundary value problems [9]. In Section 2, we shall develop this powerful technique for the boundary value problem (1.1), (1.2). Our formulation leads to several algorithms, and in particular for two point boundary value problems the obtained equations are entirely different than those in [9]. In Section 3, we shall discretize the potential equation and show that the invariant imbedding technique is directly applicable to resulting discrete equations.

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2. FORMULATION OF INVARIANT IMBEDDING METHOD

We partition the vector $\mathcal{U}(k)$ by setting $\mathcal{U}(k) = [\mathcal{V}(k), \mathcal{W}(k)]^T$, where $\mathcal{V}(k)$ is a $p \times 1$ vector, $\mathcal{W}(k)$ is a $q \times 1$ vector, and $p + q = n$. In general, the choice of the elements of $\mathcal{U}(k)$ which are to be $\mathcal{V}(k)$ and those which are to be $\mathcal{W}(k)$ is arbitrary, although for some problems one choice is more natural than any other. Once this setting is fixed, the difference system (1.1) can be written as

$$\mathcal{V}(k+1) = \mathcal{A}^1(k)\,\mathcal{V}(k) + \mathcal{A}^2(k)\,\mathcal{W}(k) + \mathcal{B}^1(k) \tag{2.1}$$

$$\mathcal{W}(k+1) = \mathcal{A}^3(k)\,\mathcal{V}(k) + \mathcal{A}^4(k)\,\mathcal{W}(k) + \mathcal{B}^2(k) \tag{2.2}$$

and the boundary conditions (1.2) take the form

$$\sum_{i=1}^{r} {}_1\mathcal{M}^i \mathcal{V}(k_i) + \sum_{i=1}^{r} {}_2\mathcal{M}^i \mathcal{W}(k_i) = \mathcal{L}^1, \tag{2.3}$$

$$\sum_{i=1}^{r} {}_3\mathcal{M}^i \mathcal{V}(k_i) + \sum_{i=1}^{r} {}_4\mathcal{M}^i \mathcal{W}(k_i) = \mathcal{L}^2, \tag{2.4}$$

where the matrices $\mathcal{A}^1(k)$, ${}_1\mathcal{M}^i$, $1 \le i \le r$ are of order $p \times p$; $\mathcal{A}^2(k)$, ${}_2\mathcal{M}^i$, $1 \le i \le r$ of order $p \times q$; $\mathcal{A}^3(k)$, ${}_3\mathcal{M}^i$, $1 \le i \le r$ of order $q \times p$; $\mathcal{A}^4(k)$, ${}_4\mathcal{M}^i$, $1 \le i \le r$ of order $q \times q$; and the vectors $\mathcal{B}^1(k)$, $\mathcal{L}^1$ are of order $p \times 1$, and $\mathcal{B}^2(k)$, $\mathcal{L}^2$ of order $q \times 1$.

There are two possible expressions for the development of the solution from (2.1), (2.2), a direct form and an inverse form. The direct form at a fixed point $k^* \in N(k_1, k_r)$ is defined as

$$\mathcal{V}(k) = \mathcal{R}^1(k,k^*)\mathcal{W}(k) + \mathcal{R}^2(k,k^*)\mathcal{V}(k^*) + \mathcal{C}^1(k,k^*) \tag{2.5}$$

$$\mathcal{W}(k^*) = \mathcal{Q}^1(k,k^*)\mathcal{W}(k) + \mathcal{Q}^2(k,k^*)\mathcal{V}(k^*) + \mathcal{C}^2(k,k^*), \tag{2.6}$$

where the matrices $\mathcal{R}^1(k,k^*)$, $\mathcal{R}^2(k,k^*)$, $\mathcal{Q}^1(k,k^*)$, $\mathcal{Q}^2(k,k^*)$ are of orders $p \times q$, $p \times p$, $q \times q$ and $q \times p$ respectively; and the vectors $\mathcal{C}^1(k,k^*)$, $\mathcal{C}^2(k,k^*)$, are of orders $p \times 1$, $q \times 1$ respectively.

Relation (2.5) is the same as

$$\mathcal{V}(k+1) = \mathcal{R}^1(k+1,k^*)\mathcal{W}(k+1) + \mathcal{R}^2(k+1,k^*)\mathcal{V}(k^*) + \mathcal{C}^1(k+1,k^*). \tag{2.7}$$

Using (2.1), (2.2) and (2.5) in (2.7), we get

$$\begin{aligned}
&[\mathcal{A}^1(k) - \mathcal{R}^1(k+1,k^*)\mathcal{A}^3(k) - \mathcal{R}^2(k+1,k^*)(\mathcal{R}^2(k,k^*))^{-1}]\mathcal{V}(k)\\
&\quad + [\mathcal{A}^2(k) - \mathcal{R}^1(k+1,k^*)\mathcal{A}^4(k) + \mathcal{R}^2(k+1,k^*)(\mathcal{R}^2(k,k^*))^{-1}\mathcal{R}^1(k,k^*)]\mathcal{W}(k)\\
&\quad + [\mathcal{B}^1(k) - \mathcal{R}^1(k+1,k^*)\mathcal{B}^2(k) + \mathcal{R}^2(k+1,k^*)(\mathcal{R}^2(k,k^*))^{-1}\mathcal{C}^1(k,k^*)\\
&\quad - \mathcal{C}^1(k+1,k^*)] = 0.
\end{aligned}$$

Thus, for $k \ge k^*$ the following system must be satisfied

$$\begin{aligned}
\mathcal{R}^1(k+1,k^*) &= [\mathcal{A}^2(k) + \mathcal{A}^1(k)\mathcal{R}^1(k,k^*)][\mathcal{A}^4(k) + \mathcal{A}^3(k)\mathcal{R}^1(k,k^*)]^{-1}\\
\mathcal{R}^2(k+1,k^*) &= [\mathcal{A}^1(k) - \mathcal{R}^1(k+1,k^*)\mathcal{A}^3(k)]\mathcal{R}^2(k,k^*)\\
\mathcal{C}^1(k+1,k^*) &= \mathcal{A}^1(k)\mathcal{C}^1(k,k^*) - \mathcal{R}^1(k+1,k^*)[\mathcal{B}^2(k) + \mathcal{A}^3(k)\mathcal{C}^1(k,k^*)] + \mathcal{B}^1(k).
\end{aligned} \tag{2.8}$$

For $k \le k^*$, (2.8) can be conveniently written as

$$\begin{aligned}
\mathcal{R}^1(k,k^*) &= [\mathcal{R}^1(k+1,k^*)\mathcal{A}^3(k) - \mathcal{A}^1(k)]^{-1}[\mathcal{A}^2(k) - \mathcal{R}^1(k+1,k^*)\mathcal{A}^4(k)]\\
\mathcal{R}^2(k,k^*) &= -[\mathcal{R}^1(k+1,k^*)\mathcal{A}^3(k) - \mathcal{A}^1(k)]^{-1}\mathcal{R}^2(k+1,k^*)\\
\mathcal{C}^1(k,k^*) &= [\mathcal{R}^1(k+1,k^*)\mathcal{A}^3(k) - \mathcal{A}^1(k)]^{-1}\\
&\qquad [\mathcal{B}^1(k) - \mathcal{R}^1(k+1,k^*)\mathcal{B}^2(k) - \mathcal{C}^1(k+1,k^*)].
\end{aligned} \tag{2.9}$$

The initial conditions for the system (2.8) as well as (2.9) are obtained from the relation (2.5) and appear as

$$\mathcal{R}^1(k^*,k^*) = 0, \quad \mathcal{R}^2(k^*,k^*) = I, \quad \mathcal{C}^1(k^*,k^*) = 0. \tag{2.10}$$

Similarly, from (2.6) and (2.2), (2.5) we find for $k \geq k^*$ that

$$\begin{aligned}
\mathcal{Q}^1(k+1,k^*) &= \mathcal{Q}^1(k,k^*)[\mathcal{A}^4(k) + \mathcal{A}^3(k)\mathcal{R}^1(k,k^*)]^{-1} \\
\mathcal{Q}^2(k+1,k^*) &= \mathcal{Q}^2(k,k^*) - \mathcal{Q}^1(k+1,k^*)\mathcal{A}^3(k)\mathcal{R}^2(k,k^*) \\
\mathcal{C}^2(k+1,k^*) &= \mathcal{C}^2(k,k^*) - \mathcal{Q}^1(k+1,k^*)[\mathcal{B}^2(k) + \mathcal{A}^3(k)\mathcal{C}^1(k,k^*)],
\end{aligned} \tag{2.11}$$

which is for $k \leq k^*$ better written as

$$\begin{aligned}
\mathcal{Q}^1(k,k^*) &= \mathcal{Q}^1(k+1,k^*)\,[\mathcal{A}^4(k) + \mathcal{A}^3(k)\mathcal{R}^1(k,k^*)] \\
\mathcal{Q}^2(k,k^*) &= \mathcal{Q}^2(k+1,k^*) + \mathcal{Q}^1(k+1,k^*)\mathcal{A}^3(k)\mathcal{R}^2(k,k^*) \\
\mathcal{C}^2(k,k^*) &= \mathcal{C}^2(k+1,k^*) + \mathcal{Q}^1(k+1,k^*)[\mathcal{B}^2(k) + \mathcal{A}^3(k)\mathcal{C}^1(k,k^*)].
\end{aligned} \tag{2.12}$$

The initial conditions for the system (2.11) as well as (2.12) are obtained from the relation (2.6) and appear as

$$\mathcal{Q}^1(k^*,k^*) = I, \quad \mathcal{Q}^2(k^*,k^*) = 0, \quad \mathcal{C}^2(k^*,k^*) = 0. \tag{2.13}$$

Equations (2.8), (2.10), (2.11), (2.13) form a RQ forward system whereas (2.9), (2.10), (2.12), (2.13) form a RQ backward system.

The inverse form at a fixed point $k^* \in N(k_1, k_r)$ is defined as

$$\mathcal{W}(k) = \mathcal{S}^1(k,k^*)\mathcal{V}(k) + \mathcal{S}^2(k,k^*)\mathcal{W}(k^*) + \mathcal{D}^1(k,k^*) \tag{2.14}$$

$$\mathcal{V}(k^*) = \mathcal{T}^1(k,k^*)\mathcal{V}(k) + \mathcal{T}^2(k,k^*)\mathcal{W}(k^*) + \mathcal{D}^2(k,k^*), \tag{2.15}$$

where the matrices $\mathcal{S}^1(k,k^*)$, $\mathcal{S}^2(k,k^*)$, $\mathcal{T}^1(k,k^*)$, $\mathcal{T}^2(k,k^*)$ are of orders $q \times p$, $q \times q$, $p \times p$ and $p \times q$ respectively; and the vectors $\mathcal{D}^1(k,k^*)$, $\mathcal{D}^2(k,k^*)$ are of orders $q \times 1$, $p \times 1$ respectively.

As above from (2.14) and (2.1), (2.2) for $k \geq k^*$, we obtain the system

$$\begin{aligned}
\mathcal{S}^1(k+1,k^*) &= [\mathcal{A}^3(k) + \mathcal{A}^4(k)\mathcal{S}^1(k,k^*)][\mathcal{A}^1(k) + \mathcal{A}^2(k)\mathcal{S}^1(k,k^*)]^{-1} \\
\mathcal{S}^2(k+1,k^*) &= [\mathcal{A}^4(k) - \mathcal{S}^1(k+1,k^*)\mathcal{A}^2(k)]\mathcal{S}^2(k,k^*) \\
\mathcal{D}^1(k+1,k^*) &= \mathcal{A}^4(k)\mathcal{D}^1(k,k^*) - \mathcal{S}^1(k+1,k^*)[\mathcal{B}^1(k) + \mathcal{A}^2(k)\mathcal{D}^1(k,k^*)] + \mathcal{B}^2(k),
\end{aligned} \tag{2.16}$$

which is for $k \leq k^*$ written as

$$\begin{aligned}
\mathcal{S}^1(k,k^*) &= [\mathcal{S}^1(k+1,k^*)\mathcal{A}^2(k) - \mathcal{A}^4(k)]^{-1}[\mathcal{A}^3(k) - \mathcal{S}^1(k+1,k^*)\mathcal{A}^1(k)] \\
\mathcal{S}^2(k,k^*) &= -[\mathcal{S}^1(k+1,k^*)\mathcal{A}^2(k) - \mathcal{A}^4(k)]^{-1}\mathcal{S}^2(k+1,k^*) \\
\mathcal{D}^1(k,k^*) &= [\mathcal{S}^1(k+1,k^*)\mathcal{A}^2(k) - \mathcal{A}^4(k)]^{-1}[\mathcal{B}^2(k) \\
&\quad - \mathcal{S}^1(k+1,k^*)\mathcal{B}^1(k) - \mathcal{D}^1(k+1,k^*)].
\end{aligned} \tag{2.17}$$

The initial conditions for the system (2.16) as well as (2.17) are obtained from the relation (2.14) and appear as

$$\mathcal{S}^1(k^*,k^*) = 0, \quad \mathcal{S}^2(k^*,k^*) = I, \quad \mathcal{D}^1(k^*,k^*) = 0. \tag{2.18}$$

Finally, from (2.15) and (2.1), (2.14) for $k \geq k^*$, we find the system

$$\begin{aligned}
\mathcal{T}^1(k+1,k^*) &= \mathcal{T}^1(k,k^*)[\mathcal{A}^1(k) + \mathcal{A}^2(k)\mathcal{S}^1(k,k^*)]^{-1} \\
\mathcal{T}^2(k+1,k^*) &= \mathcal{T}^2(k,k^*) - \mathcal{T}^1(k+1,k^*)\mathcal{A}^2(k)\mathcal{S}^2(k,k^*) \\
\mathcal{D}^2(k+1,k^*) &= \mathcal{D}^2(k,k^*) - \mathcal{T}^1(k+1,k^*)[\mathcal{B}^1(k) + \mathcal{A}^2(k)\mathcal{D}^1(k,k^*)],
\end{aligned} \tag{2.19}$$

which is for $k \le k^*$ written as

$$\begin{aligned} \mathcal{T}^1(k,k^*) &= \mathcal{T}^1(k+1,k^*)[\mathcal{A}^1(k)+\mathcal{A}^2(k)\mathcal{S}^1(k,k^*)] \\ \mathcal{T}^2(k,k^*) &= \mathcal{T}^2(k+1,k^*)+\mathcal{T}^1(k+1,k^*)\mathcal{A}^2(k)\mathcal{S}^2(k,k^*) \\ \mathcal{D}^2(k,k^*) &= \mathcal{D}^2(k+1,k^*)+\mathcal{T}^1(k+1,k^*)[\mathcal{B}^1(k)+\mathcal{A}^2(k)\mathcal{D}^1(k,k^*)]. \end{aligned} \tag{2.20}$$

The initial conditions for the system (2.19) as well as (2.20) are obtained from the relation (2.15) and appear as

$$\mathcal{T}^1(k^*,k^*) = I, \quad \mathcal{T}^2(k^*,k^*) = 0, \quad \mathcal{D}^2(k^*,k^*) = 0. \tag{2.21}$$

Equations (2.16), (2.18), (2.19), (2.21) form a ST forward system whereas (2.17), (2.18), (2.20), (2.21) form a ST backward system.

The above formulation gives several methods to obtain the solution of the boundary value problem (1.1), (2.3), (2.4) which we list as follows:

1. RQ Forward Process.

In RQ forward system let $k^* = k_1$ and solve it for all $k \in N(k_1, k_r)$. We store all the matrices $\mathcal{R}^1(k,k_1)$, $\mathcal{R}^2(k,k_1)$, $\mathcal{Q}^1(k,k_1)$, $\mathcal{Q}^2(k,k_1)$ and the vectors $\mathcal{C}^1(k,k_1)$, $\mathcal{C}^2(k,k_1)$ for all $k \in N(k_1,k_r)$. At $k = k_i$, $2 \le i \le r$ the relations (2.5), (2.6) are

$$\mathcal{V}(k_i) = \mathcal{R}^1(k_i,k_1)\mathcal{W}(k_i)+\mathcal{R}^2(k_i,k_1)\mathcal{V}(k_1)+\mathcal{C}^1(k_i,k_1) \tag{2.22i}$$

$$\mathcal{W}(k_1) = \mathcal{Q}^1(k_i,k_1)\mathcal{W}(k_i)+\mathcal{Q}^2(k_i,k_1)\mathcal{V}(k_1)+\mathcal{C}^2(k_i,k_1). \tag{2.23i}$$

The systems (2.3), (2.4); (2.22i), (2.23i), $2 \le i \le r$ are solved for the unknowns $\mathcal{V}(k_i)$, $\mathcal{W}(k_i)$, $1 \le i \le r$. For $k \in N(k_1,\ k_r)$, $k \ne k_i$, $1 \le i \le r$ the solution is then obtained by rearranging (2.5), (2.6) so that

$$\mathcal{W}(k) = (\mathcal{Q}^1(k,k_1))^{-1}[\mathcal{W}(k_1)-\mathcal{Q}^2(k,k_1)\mathcal{V}(k_1)-\mathcal{C}^2(k,k_1)] \tag{2.24}$$

$$\mathcal{V}(k) = \mathcal{R}^1(k,k_1)\mathcal{W}(k)+\mathcal{R}^2(k,k_1)\mathcal{V}(k_1)+\mathcal{C}^1(k,k_1). \tag{2.25}$$

2. Modified RQ Forward Process.

In RQ forward process we store only the matrices and vectors required in (2.3), (2.4); (2.22i), (2.23i), $2 \le i \le r$ and that too until we solve it for $\mathcal{V}(k_i)$, $\mathcal{W}(k_i)$ for a fixed i, $1 \le i \le r$. This obtained vector $\mathcal{V}(k_i)$, $\mathcal{W}(k_i)$ is used to compute the solution $\mathcal{U}(k)$ of (1.1) for all $k \in N(k_1,k_r)$. This vector $\mathcal{U}(k)$ is the required solution of the boundary value problem (1.1), (1.2).

3. Repeated RQ Forward Process.

We should expect in some problems that RQ forward process will exhibit overflow. To cope with this situation, we can switch from the RQ forward system to "a new" RQ forward system prior to overflow, and to continue with the computation. In some problems multiple switching may be necessary. In general, one does not know where the RQ forward system will overflow before actually attempting to solve the problem. In practice, one carries out the computation and if overflow occurs, one backs up and selects a switch point, say, $a_1 \in N(k_1,k_r)$ where the solutions are still good, then one attempts to solve the problem by continuing the computation from that point.

To switch from the RQ forward system to a new RQ forward system at the switch point a_1, we consider $k^* = a_1$ in (2.5), (2.6) so that basically the RQ forward system remains the same except $k^* = a_1$ instead of k_1.

We assume that to complete the forward computation only one switching at a_1 is needed, and $k_j < a_1 < k_{j+1}$ where $1 \le j \le r-1$, but fixed. The systems (2.3), (2.4); (2.5), (2.6) at $k^* = k_1$, $k = k_i$, $2 \le i \le j$ and $k = a_1$; (2.5), (2.6) at $k^* = a_1$, $k = k_i$, $j+1 \le i \le r$ are solved for the unknowns $\mathcal{V}(k_i)$, $\mathcal{W}(k_i)$, $1 \le i \le r$ and $\mathcal{V}(a_1)$, $\mathcal{W}(a_1)$. For $k \in N(k_1,\ a_1-1)$, $k \ne k_i$, $1 \le i \le j$ the solution is obtained from (2.24), (2.25) whereas for $k \in N(a_1+1,k_r)$, $k \ne k_i$, $j+1 \le i \le r$ it is obtained from

$$\mathcal{W}(k) = (\mathcal{Q}^1(k,a_1))^{-1}[\mathcal{W}(a_1)-\mathcal{Q}^2(k,a_1)\mathcal{V}(a_1)-\mathcal{C}^2(k,a_1)] \tag{2.26}$$

$$\mathcal{V}(k) = \mathcal{R}^1(k,a_1)\mathcal{W}(k)+\mathcal{R}^2(k,a_1)\mathcal{V}(a_1)+\mathcal{C}^1(k,a_1). \tag{2.27}$$

The case where multiple switching is needed can be extended easily.

4. ST Forward Process.

In ST forward system let $k^* = k_1$ and solve it for all $k \in N(k_1, k_r)$. We store all the matrices $\mathcal{S}^1(k,k_1)$, $\mathcal{S}^2(k,k_1)$, $\mathcal{T}^1(k,k_1)$, $\mathcal{T}^2(k,k_1)$ and the vectors $\mathcal{D}^1(k,k_1)$, $\mathcal{D}^2(k,k_1)$ for all $k \in N(k_1,k_r)$. The systems (2.3), (2.4); (2.14), (2.15) at $k^* = k_1$, $k = k_i$, $2 \le i \le r$ are solved for the unknowns $\mathcal{V}(k_i)$, $\mathcal{W}(k_i)$, $1 \le i \le r$. For $k \in N(k_1,k_r)$, $k \ne k_i$, $1 \le i \le r$ the solution is then obtained by rearranging (2.14), (2.15) so that

$$\mathcal{V}(k) = (\mathcal{T}^1(k,k_1))^{-1}[\mathcal{V}(k_1) - \mathcal{T}^2(k,k_1)\mathcal{W}(k_1) - \mathcal{D}^2(k,k_1)] \tag{2.28}$$

$$\mathcal{W}(k) = \mathcal{S}^1(k,k_1)\mathcal{V}(k_1) + \mathcal{S}^2(k,k_1)\mathcal{W}(k_1) + \mathcal{D}^1(k,k_1). \tag{2.29}$$

5. Modified ST Forward Process.

In ST forward process we use the same technique as in the modified RQ forward process.

6. Repeated ST Forward Process.

As in repeated RQ forward process we switch from ST forward system to ST forward system as often as needed.

7. Repeated RQ-ST Forward Process.

We begin with RQ(ST) forward system and whenever necessary switch to ST(RQ) forward system.

1′. RQ Backward Process.

In RQ backward system let $k^* = k_r$ and solve it backward for all $k \in N(k_1,k_r)$. Rest of the technique is same as in RQ forward process.

Finally, we remark that corresponding to 2 – 7 we have 2′ – 7′ where forward is replaced by backward.

3. APPLICATIONS TO POTENTIAL EQUATION

Consider the potential equation

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \tag{3.1}$$

over the rectangle $0 \le x \le \alpha$, $0 \le y \le \beta$. We shall assume that the values of $v(x,y)$ are specified on the boundary of the rectangle. If we let $v_{k,\ell}$ denote $v(kh_1, \ell h_2)$ where $(K+1)h_1 = \alpha$, $(L+1)h_2 = \beta$, and replace $\frac{\partial^2 v}{\partial x^2}$ by $\frac{1}{h_1^2}(v_{k+1,\ell} - 2v_{k,\ell} + v_{k-1,\ell})$ and $\frac{\partial^2 v}{\partial y^2}$ by $\frac{1}{h_2^2}(v_{k,\ell+1} - 2v_{k,\ell} + v_{k,\ell-1})$ in (3.1) it takes the form

$$v_{k+1,\ell} + \lambda v_{k,\ell+1} - (2+2\lambda)v_{k,\ell} + \lambda v_{k,\ell-1} + v_{k-1,\ell} = 0; \qquad 1 \le k \le K,\ 1 \le \ell \le L \tag{3.2}$$

where $\lambda = h_1^2/h_2^2$.

Let us define the $K+2$ vectors $\mathcal{V}(k)$ of order $L \times 1$ by $\mathcal{V}(k) = (v_{k,\ell})$, $1 \le \ell \le L$. Since we are given the values $(v_{0,\ell})$ and $(v_{K+1,\ell})$, we have, say

$$\mathcal{V}(0) = \mathcal{C}, \quad \mathcal{V}(K+1) = \mathcal{D}. \tag{3.3}$$

Now we define an $L \times L$ matrix $\mathcal{Q} = (q_{ij})$, where

$$q_{ij} = \begin{cases} (2+2\lambda) & \text{if } i = j \\ -\lambda & \text{if } \mid i-j \mid = 1 \\ 0 & \text{otherwise.} \end{cases}$$

We also define vectors $\mathcal{R}(k)$ of order $L \times 1$ by $\mathcal{R}(k) = (r_{k,\ell})$, $1 \le \ell \le L$ where

$$r_{k\ell} = \begin{cases} \lambda v_{k,0} & \text{if } \ell = 1 \\ \lambda v_{k,L+1} & \text{if } \ell = L \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, the vectors $\mathcal{R}(k)$ are given by the boundary conditions on the edges $y = 0$ and $y = \beta$. With these notations, (3.2) can be written as the second order difference system

$$\mathcal{V}(k+1) - \mathcal{Q}\mathcal{V}(k) + \mathcal{V}(k-1) + \mathcal{R}(k) = 0,\ 1 \le k \le K \tag{3.4}$$

subject to the boundary conditions (3.3).

If we define $\mathcal{W}(k)$ by $\mathcal{V}(k+1) = \mathcal{W}(k)$, then (3.4) is equivalent to the first order system

$$\begin{aligned} &\mathcal{V}(k+1) = \mathcal{W}(k) \\ &\mathcal{W}(k+1) = -\mathcal{V}(k) + \mathcal{Q}\mathcal{W}(k) - \mathcal{R}(k+1),\ 0 \le k \le K-1 \end{aligned} \tag{3.5}$$

and the boundary conditions (3.3) are the same as

$$\mathcal{V}(0) = \mathcal{C}, \quad \mathcal{W}(K) = \mathcal{D}. \tag{3.6}$$

On comparing the problem (3.5), (3.6) with (2.1) – (2.4) we find that $p = q = L$, $r = 2$, $k_1 = 0$, $k_2 = K$, $\mathcal{A}^1(k) = 0$, $\mathcal{A}^2(k) = I$, $\mathcal{B}^1(k) = 0$, $\mathcal{A}^3(k) = -I$, $\mathcal{A}^4(k) = \mathcal{Q}$, $\mathcal{B}^2(k) = -\mathcal{R}(k+1)$, ${}_1\mathcal{M}^1 = I$, ${}_1\mathcal{M}^2 = {}_2\mathcal{M}^1 = {}_2\mathcal{M}^2 = 0$, $\mathcal{L}^1 = \mathcal{C}$, ${}_3\mathcal{M}^1 = {}_3\mathcal{M}^2 = {}_4\mathcal{M}^1 = 0$, ${}_4\mathcal{M}^2 = I$, $\mathcal{L}^2 = \mathcal{D}$. Therefore, all the algorithms developed in Section 2 can be applied to solve (3.5), (3.6). In particular, RQ Forward Process simply reduces to

$$\begin{aligned} &\mathcal{R}^1(k+1,0) = [\mathcal{Q} - \mathcal{R}^1(k,0)]^{-1} \\ &\mathcal{R}^2(k+1,0) = \mathcal{R}^1(k+1,0)\mathcal{R}^2(k,0) \\ &\mathcal{C}^1(k+1,0) = \mathcal{R}^1(k+1,0)[\mathcal{R}(k+1) + \mathcal{C}^1(k,0)] \\ &\quad \mathcal{R}^1(0,0) = 0, \quad \mathcal{R}^2(0,0) = I, \quad \mathcal{C}^1(0,0) = 0 \\ &\mathcal{Q}^1(k+1,0) = \mathcal{Q}^1(k,0)\mathcal{R}^1(k+1,0) \\ &\mathcal{Q}^2(k+1,0) = \mathcal{Q}^2(k,0) + \mathcal{Q}^1(k+1,0)\mathcal{R}^2(k,0) \\ &\mathcal{C}^2(k+1,0) = \mathcal{C}^2(k,0) + \mathcal{Q}^1(k+1,0)[\mathcal{R}(k+1) + \mathcal{C}^1(k,0)] \\ &\quad \mathcal{Q}^1(0,0) = I, \quad \mathcal{Q}^2(0,0) = 0, \quad \mathcal{C}^2(0,0) = 0 \end{aligned}$$

and

$$\begin{aligned} \mathcal{V}(K) &= \mathcal{R}^1(K,0)\mathcal{D} + \mathcal{R}^2(K,0)\mathcal{C} + \mathcal{C}^1(K,0) \\ \mathcal{W}(0) &= \mathcal{Q}^1(K,0)\mathcal{D} + \mathcal{Q}^2(K,0)\mathcal{C} + \mathcal{C}^2(K,0) \end{aligned}$$

and finally

$$\begin{aligned} \mathcal{W}(k) &= (\mathcal{Q}^1(k,0))^{-1}[\mathcal{W}(0) - \mathcal{Q}^2(k,0)\mathcal{C} - \mathcal{C}^2(k,0)] \\ \mathcal{V}(k) &= \mathcal{R}^1(k,0)\mathcal{W}(k) + \mathcal{R}^2(k,0)\mathcal{C} + \mathcal{C}^1(k,0). \end{aligned}$$

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